Brane-world inflation without inflaton on the brane

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Inspired by the Randall-Sundrum brane-world scenario, we investigate the possibility of brane-world inflation driven not by an inflaton field on the brane, but by a bulk, dilaton-like gravitational field. As a toy model for the dilaton-like gravitational field, we consider a minimally coupled massive scalar field in the bulk 5-dimensional spacetime, and look for a perturbative solution in the anti-de Sitter (AdS) background. For an adequate range of the scalar field mass, we find a unique solution that has non-trivial dependence on the 5th dimensional coordinate and that induces slow-roll inflation on the brane.

I. INTRODUCTION

It is now widely accepted that our spacetime is not 4-dimensional but higher dimensional from the unified theoretical point of view. Horava and Witten showed the possibility that desirable gauge fields may consistently appear on the 10-dimensional boundary of Z_2-symmetric 11-dimensional spacetime [1]. A low-energy, 5-dimensional realization of the Horava-Witten theory was first discussed by Lukas et al. [2]. They also analyzed brane-world cosmology in the context of this 5-dimensional theory [3,4].

Recently, Randall and Sundrum wrote two very interesting papers [5,6] on another possible low-energy realization of the Horava-Witten theory. In [5], they found an interesting Z_2-symmetric solution of the 5-dimensional Einstein equations with a negative cosmological constant. In this solution, two boundary branes with positive and negative tensions are embedded in the 5-dimensional anti-de Sitter (AdS) space and the tensions of the branes are chosen so that the effective cosmological constant on the branes vanishes and the 4-dimensional Minkowski space is realized on the branes. They then showed that the mass-hierarchy problem in particle physics may be solved if we live on the negative tension brane. This work has received much attention from the particle physics community, and subsequently a large number of papers have been published on it. However, it was soon realized that there exists the so-called radion mode that describes fluctuations of the distance between the two branes, and this mode causes an unacceptable modification of the effective gravitational theory on the negative tension brane, unless the radion is stabilized rather artificially [7–9].

In the second paper [6], Randall and Sundrum showed that the negative tension brane may actually be absent if we live on the positive tension brane. They found because of the curvature of AdS, even if the extra-dimension is infinite, gravity on the brane is nicely confined around the brane and the Einstein gravity is effectively recovered on the brane [6,7,10,11]. As a result, although the hierarchy problem remains unsolved, this model has attracted much attention from the relativity/cosmology community, and the brane-world cosmology has boomed [12–29].

To name a few, the Friedmann equation on the brane is discussed in [14–16,24–26,29]. The quantum creation of a brane-world is discussed in [26,27]. The formulation and evolution of cosmological perturbations in the brane-world are discussed in [18–20,23,25]. And inflation on the brane is discussed by many people [26–28,30–38].

In almost all of these works, the 5-dimensional bulk spacetime is assumed to be vacuum except for the presence of the cosmological constant, and the matter fields on the brane are regarded as responsible for the dynamics of the brane. However, from the unified theoretic point of view, the gravitational action is not necessarily the Einstein-Hilbert action. In fact, string theory tells us that the dimensionally reduced effective action includes not only higher-order curvature terms but also dilatonic gravitational scalar fields. Thus at the level of the "low-energy" 5-dimensional theory, it is naturally expected that there appears a dilaton-like scalar field in addition to the Einstein-Hilbert action [2]. Thus it is of interest to investigate how such a scalar field in the 5-dimensional theory affects the brane-world. In this connection, very recently Maeda and Wands have investigated dilaton-gravity in the brane-world scenario [39].

In this paper, we investigate whether it is possible or not to have inflation on the brane solely by a bulk gravitational scalar field. Thus we consider a scenario of inflation on the brane without introducing an inflaton field on the brane.‡ A similar model of inflation in 5-dimensional dilatonic gravity was discussed recently by Nojiri, Obregon and Odintsov

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‡This type of inflation is called "modular inflation" in [4].

[40]. We model the dilaton-like bulk gravitational scalar by a minimally coupled scalar field. It is known that a scalar-tensor gravitational theory is conformally equivalent to the Einstein theory plus a minimally coupled scalar field [41]. So our model may be regarded as a conformally transformed scalar-tensor gravitational theory.

Very recently, the quantum fluctuations in a similar brane-world scenario has been discussed by Kobayashi, Koyama and Soda [42]. They have shown that the extra-dimensional corrections due to the so-called Kaluza-Klein modes are small and the scenario is observationally acceptable just as the ordinary 4-dimensional scenario of inflation.

This paper is organized as follows. In Sec. II, we first recapitulate the effective 4-dimensional gravitational equations on the brane and derive the Friedmann equation on the brane under the presence of a bulk scalar field. Then we discuss general conditions for inflation to occur on the brane by the bulk scalar field. In Sec. III, we solve the Einstein-scalar field equations in the bulk perturbatively, and give explicit solutions that give rise to inflation on the brane. Finally, we summarize our results and discuss the implications in Sec. IV.

II. EFFECTIVE 4-DIMENSIONAL EQUATIONS ON THE BRANE

In order to understand the general conditions required to induce inflation on the brane, we review the effective 4-dimensional gravitational equations, and derive the Friedmann equation under the presence of a bulk scalar field.

A. The Einstein equations on the brane

The general form of the effective gravitational equations on the brane was discussed in [10]. Following it, we consider the metric near the brane in the Gaussian normal coordinates,

$$ds^2 = g_{ab}dx^a dx^b = d\chi^2 + q_{\mu\nu}dx^\mu dx^\nu \,, \tag{2.1}$$

where the brane is located at $\chi =$ constant. Without loss of generality, we set this constant at zero. We assume that the 5-dimensional bulk gravitational equation takes the form,

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda_5 g_{ab} = \kappa_5^2 \left(T_{ab} + S_{ab}\delta(\chi)\right) \,, \tag{2.2}$$

where the 5-dimensional energy-momentum tensor T_{ab} stands for a dilaton-like scalar field. For our toy model, we assume it is given by a minimally coupled scalar field,

$$T_{ab} = \phi_{,a}\phi_{,b} - g_{ab}\left(\frac{1}{2}g^{cd}\phi_{,c}\phi_{,d} + V(\phi)\right). \tag{2.3}$$

The energy-momentum tensor on the brane S_{ab} is assumed to be of the form,

$$S_{ab} = -\sigma q_{ab} + \tau_{ab} \,; \quad \tau_{ab}n^b = 0 \,, \tag{2.4}$$

where n^a is the vector unit normal to the brane, σ is the brane tension and τ_{ab} describes the 4-dimensional matter fields. The field equation for the bulk scalar ϕ is

$$\Box_{(5)}\phi - V'(\phi) = \partial_\chi^2 \phi + \Box_{(4)}\phi - V'(\phi) = 0. \tag{2.5}$$

According to [10], the induced 4-dimensional Einstein tensor $^{(4)}G_{ab}$ on the $\chi =$ constant $\neq 0$ hypersurface is given by

$$^{(4)}G_{ab} = \frac{2\kappa_5^2}{3}\left[(T_{cd} - \frac{\Lambda_5}{\kappa_5^2}g_{cd})q_a{}^c q_b{}^d + \left((T_{cd} - \frac{\Lambda_5}{\kappa_5^2}g_{cd})n^c n^d - \frac{1}{4}(T^c{}_c - \frac{5\Lambda_5}{\kappa_5^2})\right)q_{ab}\right]$$
$$-E_{ab} + KK_{ab} - K_a{}^c K_{bc} - \frac{1}{2}q_{ab}(K^2 - K^{cd}K_{cd}), \tag{2.6}$$

where $K_{ab} = q_a{}^c n_{c;d}q^d{}_b$ is the extrinsic curvature of the $\chi =$ constant hypersurface, and $E_{ab} = C_{acbd}n^c n^d$ where C_{acbd} is the 5-dimensional Weyl tensor.

Following the spirit of the RS brane-world, we assume the bulk spacetime is Z_2 (reflection) symmetric with respect to the brane. This implies the junction condition,

$$[K_{\mu\nu}] = 2K_{\mu\nu}^+ = -2K_{\mu\nu}^- = -\kappa_5^2\left(\frac{\sigma}{3}q_{\mu\nu} + \tau_{\mu\nu} - \frac{1}{3}q_{\mu\nu}\tau\right) \,, \tag{2.7}$$

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where $[K_{\mu\nu}] = K_{\mu\nu}^+ - K_{\mu\nu}^-$. Substituting Eq.(2.3) into Eq.(2.6), and taking the limit $\chi \to \pm 0$, $^{(4)}G_{ab}$ on the brane becomes

$$G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa_4^2(T_{\mu\nu}^{(s)} + \tau_{\mu\nu}) + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}, \tag{2.8}$$

where

$$\Lambda_4 = \frac{1}{2}\left(\Lambda_5 + \frac{1}{6}\kappa_5^4 \sigma^2\right), \tag{2.9}$$

$$\kappa_4^2 = \frac{\kappa_5^4 \sigma}{6}, \tag{2.10}$$

$$T_{\mu\nu}^{(s)} = \frac{1}{\kappa_5^2 \sigma}\left(4\phi_{,\mu}\phi_{,\nu} + \left(\frac{3}{2}(\phi_{,\chi})^2 - \frac{5}{2}q^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - 3V(\phi)\right)q_{\mu\nu}\right), \tag{2.11}$$

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau_\nu{}^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}q_{\mu\nu}\tau^2. \tag{2.12}$$

B. The Friedmann equation on the brane

We consider a spatially isotropic, homogeneous universe on the brane:

$$ds^2\big|_{\chi=0} = q_{\mu\nu}(\chi = 0)dx^\mu dx^\nu = -dt^2 + a(t)^2 \gamma_{ij}dx^i dx^j, \tag{2.13}$$

where γ_{ij} is the metric of a constant curvature space with curvature $K = \pm 1, 0$. We assume the matter fields on the brane satisfy the energy-momentum conservation law $D_\nu \tau^{\mu\nu} = 0$, where D_ν is the covariant derivative with respect to $q_{\mu\nu}$. This implies [10]

$$D_\nu \tau_\mu{}^\nu \propto D_\nu K_\mu{}^\nu - D_\mu K = \kappa_5^2 T_{ab} n^a q^b{}_\mu = \kappa_5^2 T_{\chi\mu} = 0. \tag{2.14}$$

Hence, the bulk scalar ϕ is even with respect to χ, i.e., $\phi_{,\chi}|_{\chi=0} = 0$.

We assume the perfect fluid form for $\tau_{\mu\nu}$ on the brane:

$$\tau^{\mu\nu} = \rho^{(m)}t^\mu t^\nu + Ph^{\mu\nu}. \tag{2.15}$$

Then, the Friedmann equation on the brane becomes

$$3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] = \kappa_4^2(\rho^{(s)} + \rho^{(m)}) + \frac{\kappa_5^4}{12}\rho^{(m)2} - E_{tt} + \Lambda_4, \tag{2.16}$$

where

$$\rho^{(s)} = \frac{3}{\kappa_5^2 \sigma}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) = \frac{1}{2}\dot{\Phi}^2 + \tilde{V}(\Phi). \tag{2.17}$$

Here, we have introduced the effective scalar field Φ on the brane by rescaling the bulk scalar field as

$$\Phi := \sqrt{\frac{3}{\kappa_5^2 \sigma}}\phi. \tag{2.18}$$

Now we examine if there is a situation in which all but the $\rho^{(s)}$ term on the right-hand-side of Eq. (2.16) can be neglected, and inflation occurs due to the potential $\tilde{V}(\Phi)$. For simplicity, we neglect the matter terms; their role is the same as the standard 4-dimensional theory except for the presence of the term quadratic in $\rho^{(m)}$. As for the Λ_4 term, we assume it is fine-tuned to a very small value or it can be just absorbed into the $\tilde{V}(\Phi)$ term. In any case, the cosmological constant problem is beyond the scope of this paper. Then, the remaining, possibly dangerous term is the E_{tt} term. Since it carries information of the bulk gravitational field, it cannot be determined solely by the 4-dimensional equations. Nevertheless, it is possible to obtain some general features of the term by considering the Bianchi identities.

As discussed in [10], the spatial homogeneity of the brane implies

$$D_\mu \pi^{\mu\nu} = 0. \tag{2.19}$$

Then, the 4-dimensional Bianchi identities imply

$$\kappa_4^2 D^\mu T_{\mu\nu}^{(s)} = D^\mu E_{\mu\nu}. \tag{2.20}$$

Note that $E_{\mu\nu}$ is traceless i.e. $E_t^t = -E_i^i$. The only non-trivial component of the above equation in the present case is the time-component, which becomes

$$\frac{\kappa_5^2}{6}(\ddot{\phi} - 4\,\partial_\chi^2\,\phi + V')\dot{\phi} = -\frac{1}{a^4}\partial_t(a^4 E_{tt}). \tag{2.21}$$

Using the 5-dimensional field equation (2.5), this is rewritten as

$$\frac{\kappa_5^2}{2}(\partial_\chi^2\phi + \frac{\dot{a}}{a}\dot{\phi})\dot{\phi} = \frac{1}{a^4}\partial_t(a^4 E_{tt}). \tag{2.22}$$

Therefore

$$E_{tt} = \frac{\kappa_5^2}{2a^4}\int^t a^4\dot{\phi}(\partial_\chi^2\phi + \frac{\dot{a}}{a}\dot{\phi})\,dt. \tag{2.23}$$

The integration constant in the above integral gives rise to the 'dark radiation' term proportional to a^{-4}. Hence we may neglect it if inflation should occur. We then see that E_{tt} can be neglected if both $\dot{\phi}$ and $\partial_\chi^2\phi$ are sufficiently small, i.e.,

$$\dot{\phi}^2 \ll V(\phi)\,, \quad |\partial_\chi^2\phi| \leq \frac{\dot{a}}{a}|\dot{\phi}|\,, \tag{2.24}$$

on the brane.

Thus a sufficient condition for inflation to occur on the brane is that ϕ is a slowly varying function with respect to both t and χ in the vicinity of the brane. In the next section, we look for a solution with such a property.

III. THE SOLUTION IN THE BULK

We want to find a solution of the field equations that has non-trivial dynamics in the bulk and gives rise to inflation on the brane. Since such a solution will naturally have non-trivial dependence on t and χ, we take a perturbative approach.

A. Model

It has been pointed out in [26] that the metric for a natural brane universe model in the cosmological context takes the form,

$$ds^2 = dr^2 + (H\ell)^2 \sinh^2(r/\ell)[-dt^2 + H^{-2}\cosh^2(Ht)d\Omega_{(3)}^2] \quad (r \leq r_0), \tag{3.1}$$

where $d\Omega_{(3)}^2$ is the metric on the unit 3-sphere, r_0 is the location of the brane and the brane asymptotically inflates with the Hubble rate H,

$$H(\ell) = \frac{1}{\ell \sinh(r_0/\ell)}\,; \quad \ell = \left|\frac{6}{\Lambda_5}\right|^{1/2}. \tag{3.2}$$

The brane tension σ is given by

$$\sigma = \sigma_c \coth(r_0/\ell)\,; \quad \sigma_c = \frac{6}{\kappa_5^2\ell}\,, \tag{3.3}$$

where σ_c is the critical tension that gives $\Lambda_4 = 0$ and reproduces the original RS brane-world [6]. A nice feature of this model is that it can be interpreted as the brane universe created from nothing, and the universe naturally inflates after creation.

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Here we essentially adopt this model, but slightly modify the interpretation of the model parameters and consider the following scenario. We assume that the 5-d cosmological constant Λ_5 and the brane tension σ are determined from some yet unknown unified theory such that $\sigma = \sigma_c$. However, due to dynamics of the bulk gravitational field, the dilaton-like scalar field with an effective potential $V(\phi)$ appears in the bulk. We assume $V > 0$ which may vary very slowly in space and time. The brane-world is created from nothing in this situation. Then, as will be discussed in detail below, inflation occurs on the brane. As the universe evolves, the potential V decreases and eventually becomes zero (or very small) and the standard Friedmann universe is recovered in the low energy limit.

The specific model we consider is as follows. For simplicity, we assume the potential of the form,

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2, \tag{3.4}$$

with $m^2 < 0$ and consider the situation when ϕ is sufficiently close to zero. In the lowest order, we put $\phi = 0$. Then the 5-dimensional bulk spacetime is AdS with the cosmological constant $\Lambda_{5,eff}$, where

$$\Lambda_{5,eff} = \Lambda_5 + \kappa_5^2 V_0. \tag{3.5}$$

The Friedmann equation on the brane becomes

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = H^2 = \frac{\kappa_5^2 V_0}{6}, \tag{3.6}$$

where $K = 1$. Note that V_0 must satisfy the condition $|\Lambda_5| > \kappa_5^2 V_0$ so that the background remains effectively AdS, i.e., $\Lambda_{5,eff} < 0$. Thus the lowest order bulk spacetime has the metric (3.1) with ℓ replaced by ℓ_{eff}, where

$$\ell_{eff}^2 = \frac{6}{|\Lambda_{5,eff}|}. \tag{3.7}$$

On this effective AdS background, we look for a perturbative solution for ϕ. Assuming the solution is spherically symmetric, the field equation in the bulk becomes

$$\frac{1}{H^2 l_{eff}^2 \sinh^2(r/l_{eff}) \cosh^3(Ht)} \partial_t \left(\cosh^3(Ht)\partial_t\phi\right) - \frac{1}{\sinh^4(r/l_{eff})} \partial_r \left(\sinh^4(r/l_{eff})\partial_r\phi\right) + m^2\phi = 0, \tag{3.8}$$

with the boundary condition $\partial_r\phi = 0$ at $r = r_0$. Once the solution is found, we can solve the Friedmann equation (2.16) together with the equation (2.23) for E_{tt} with the identification $\chi = r - r_0$, to find the cosmological evolution on the brane perturbatively.

B. Slow-roll condition

Our task is to find a regular solution of Eq. (3.8) that satisfies the slow-roll condition on the brane. From Eq.(3.7),

$$H^2\ell_{eff}^2 = \frac{1}{\sinh^2(r_0/l_{eff})} = \frac{1}{z_0^2 - 1}, \tag{3.9}$$

where, $z_0 = \cosh(r_0/l_{eff})$. Using Eqs. (3.5) and (3.6), the above equation can be expressed as

$$H^2\ell_{eff}^2 = \frac{\kappa_5^2 V_0}{|\Lambda_5| - \kappa_5^2 V_0} = \frac{1}{|\Lambda_5|/\kappa_5^2 V_0 - 1}. \tag{3.10}$$

Hence the position of brane is determined by the ratio between the 5-dimensional cosmological constant Λ_5 and the vacuum energy V_0,

$$z_0^2 = \cosh^2(r_0/\ell_{eff}) = \frac{|\Lambda_5|}{\kappa_5^2 V_0} > 1. \tag{3.11}$$

Note that the last inequality is the condition for the bulk spacetime to be AdS. From above the relation, we find

$$\frac{|m^2|}{H^2} = |m^2|l_{eff}^2(z_0^2 - 1). \tag{3.12}$$

We expect slow-roll inflation to occur if $|m^2|/H^2 \ll 1$.

C. Solution

To solve the field equation (3.8), we look for a solution of the separable form,

$$\phi(t,r) = \psi(t)\,u(r)\,. \tag{3.13}$$

Then we obtain

$$\left[-\frac{1}{\sinh^4(r/l_{eff})}\partial_r \sinh^4(r/l_{eff})\partial_r + m^2 + \frac{\lambda^2}{l_{eff}^2 \sinh^2(r/l_{eff})} \right] u(r) = 0, \tag{3.14}$$

$$\left[\frac{1}{\cosh^3(Ht)}\partial_t \cosh^3(Ht)\partial_t - H^2\lambda^2 \right] \psi(t) = 0, \tag{3.15}$$

where λ^2 is the separation constant. The general solutions of Eqs. (3.14) and (3.15) are given by

$$u = u_\gamma(r) = A\frac{P_{\nu-1/2}^{-\gamma-3/2}(\cosh(r/\ell_{eff}))}{\sinh^{3/2}(r/\ell_{eff})} + B\frac{Q_{\nu-1/2}^{-\gamma-3/2}(\cosh(r/\ell_{eff}))}{\sinh^{3/2}(r/\ell_{eff})}, \tag{3.16}$$

$$\psi = \psi_\gamma(t) = C\frac{P_{1/2}^{\mu}(\tanh(Ht))}{\cosh^{3/2}(Ht)} + D\frac{P_{1/2}^{-\mu}(\tanh(Ht))}{\cosh^{3/2}(Ht)}, \tag{3.17}$$

with $\mu := \sqrt{9/4 + \lambda_\gamma^2}$, $\nu := \sqrt{m^2 l_{eff}^2 + 4}$ and $\lambda^2 = \lambda_\gamma^2 := \gamma(\gamma + 3)$. The relative magnitude of the coefficients A and B are to be determined by the boundary conditions. The coefficients C and D are to be determined by the initial condition. A natural initial condition in the scenario of creation of the brane universe [26] would be to require $\dot{\psi}_\gamma = 0$ at $t = 0$. However, here we leave the initial condition unspecified.

First, we consider the eigen-function $u_\gamma(r)$. The boundary conditions to be satisfied are

$$\partial_r u_\gamma|_{r=r_0} = 0 \quad (Z_2\text{-symmetry}), \tag{3.18}$$

$$u_\gamma|_{r=0} = 0 \quad (\text{Regularity at the origin}). \tag{3.19}$$

Because $Q_\alpha^\beta(z)$ is singular as $z \to 1$, the regularity condition at the origin $r = 0$ implies $B = 0$. In addition, $P_\alpha^\beta(z)$ behaves as $(z-1)^{-\beta/2}$ in the limit $z \to 1$. Hence we must have $\gamma > 0$, i.e., $\lambda_\gamma^2 > 0$. On the other hand, from the asymptotic form of $P_\alpha^\beta(z)$ for $z \gg 1$, corresponding to $r \to \infty$, we find

$$\frac{P_{\nu-1/2}^{-\gamma-3/2}(\cosh(r/\ell_{eff}))}{\sinh^{3/2}(r/\ell_{eff})} \to z^{-\nu-2} + z^{\nu-2}, \tag{3.20}$$

where $z = \cosh(r/\ell_{eff})$. For $|\nu| < 2$ ($m^2 < 0$) which we assume, the solution is damped as $r \to \infty$. Thus it is possible for the solution to have at least one extremum where $\partial_r u_\gamma = 0$. Therefore, the eigen-function u_γ is given by

$$u_\gamma(r) = \frac{P_{\nu-1/2}^{-\gamma-3/2}(\cosh(r/\ell_{eff}))}{\sinh^{3/2}(r/\ell_{eff})}\,. \tag{3.21}$$

Given the location of the brane, the eigen-value γ is determined by the boundary condition at the brane,

$$(\nu + 2)z_0 P_{\nu-1/2}^{-\gamma-3/2}(z_0) = (\nu + \gamma + 2)P_{\nu+1/2}^{-\gamma-3/2}(z_0)\,. \tag{3.22}$$

Some examples of the behavior of $u_\gamma(r)$ are shown in Fig. 1. In extreme cases of the model parameters, the above equation can be solved analytically. We will come back to this issue shortly.
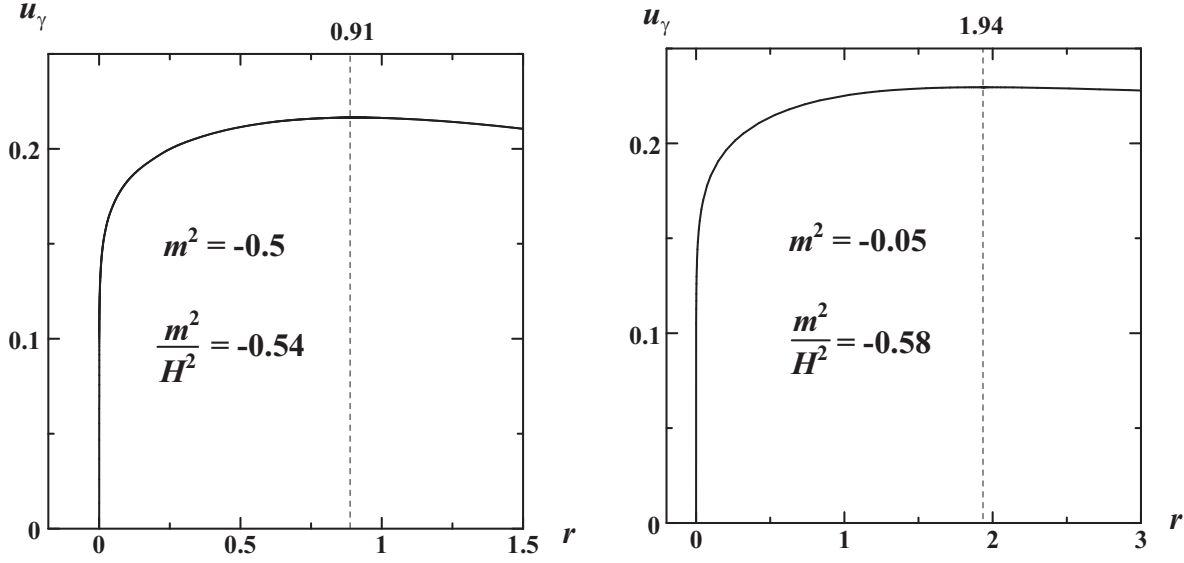
Fig. 1   Radial functions for a couple of model parameters in units of $\ell_{eff} = 1$. The vertical scale is arbitrary. The dotted vertical line indicates the location of the brane. In both cases, the eigen-value is $\gamma = 0.1$.

Next, we consider the time-function $\psi_\gamma(t)$. For $Ht \gg 1$, we have

$$\frac{P_{1/2}^{\pm\mu}(\tanh(Ht))}{\cosh^{3/2}(Ht)} \propto \exp\left[\left(\pm\mu - \frac{3}{2}\right)Ht\right], \tag{3.23}$$

Noting that $\mu = \sqrt{9/4 + \lambda_\gamma^2}$ and the condition $\lambda_\gamma^2 > 0$, we find the solution approaches asymptotically to

$$\psi_\gamma(t) \to C\frac{P_{1/2}^{\mu}(\tanh(Ht))}{\cosh^{3/2}(Ht)} \propto \exp\left[\left(\mu - \frac{3}{2}\right)Ht\right]. \tag{3.24}$$

Thus slow-rolling occurs if the eigen-value λ_γ^2 satisfies the condition

$$0 < \lambda_\gamma^2 \ll 1 \Longleftrightarrow 0 < \gamma \ll 1. \tag{3.25}$$

We have found numerically that there exists a solution that satisfies the above condition for several examples of the model parameters satisfying $|m^2|/H^2 \ll 1$ (see Fig. 1 for a couple of examples). To show analytically the existence of such a solution, let us consider a couple of cases with extreme values of the model parameters.

First, we consider the case when the vacuum energy V_0 almost cancels out the cosmological constant Λ_5. Namely,

$$\left|\frac{\Lambda_{5,eff}}{\Lambda_5}\right| = \frac{|\Lambda_5 + \kappa_5^2 V_0|}{|\Lambda_5|} = z_0^2 - 1 \ll 1. \tag{3.26}$$

Then the effective curvature radius ℓ_{eff} becomes very large compared with ℓ. In this case, Eq. (3.22) gives

$$z_0 = z_{0,s} + O((z_0 - 1)^2), \tag{3.27}$$

where

$$z_{0,s} = 1 + \frac{4\gamma(5 + 2\gamma)}{32 + 13\gamma - 4(2 + \gamma)\nu^2}. \tag{3.28}$$

Therefore, using Eq. (3.12), we find

$$\frac{|m^2|}{H^2} = 5\gamma + \frac{(49 + 4\nu^2)\gamma^2}{32 - 8\nu^2} + O(\gamma^3). \tag{3.29}$$

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Thus in the leading order we obtain

$$\gamma = \frac{|m^2|}{5H^2}.$$ (3.30)

Hence, provided $|m^2|/H^2 \ll 1$, the slow-roll condition (3.25) is satisfied.

Next, we consider the opposite case when V_0 is very small; $\kappa_5^2 V_0/|\Lambda_5| \ll 1$. In this case, ℓ_{eff} is approximately equal to ℓ. The condition $|m^2|/H^2 \ll 1$ implies $z_0 = \cosh(r_0/\ell_{eff}) \gg 1$ and $|m^2|\ell_{eff}^2 \ll |m^2|/H^2$. Then Eq. (3.22) is solved to give

$$z_0^2 = \frac{(\nu + \gamma + 1)(\nu^2 + (\nu - 4)\gamma - 4)}{4(\nu^2 - 3\nu + 2)},$$ (3.31)

where $\nu = \sqrt{4 + m^2\ell_{eff}^2} \approx 2 + m^2\ell_{eff}^2/4$. Therefore, in the leading order we obtain

$$\gamma = \frac{|m^2|}{6H^2} - \frac{|m^2|\ell_{eff}^2}{3}.$$ (3.32)

Thus the slow-roll condition (3.25) is satisfied also in this case if $|m^2|/H^2 \ll 1$.

Although we have no analytical proof, the above results, together with some numerical examples we have checked, strongly indicate that the slow-roll solution exists for all possible values of $\kappa_5^2 V_0$ provided $|m^2|/H^2 \ll 1$.

Finally, for completeness, we evaluate the $\partial_\chi^2 \phi$ term in E_{tt} given by Eq. (2.23). Noting that $\chi = r - r_0$ in the present model, we calculate $H^{-2}\partial_r^2\phi/\phi$ on the brane. Using Eq. (3.22) and the following recursion formula,

$$(\alpha - \beta + 1)P_{\alpha+1}^\beta(z) - (2\alpha + 1)zP_\alpha^\beta(z) + (\alpha + \beta)P_{\alpha-1}^\beta(z) = 0,$$ (3.33)

it is expressed exactly as

$$\frac{\partial_r^2\phi}{H^2\phi} = \lambda_\gamma^2 - \frac{|m^2|}{H^2}.$$ (3.34)

Hence E_{tt} can be consistently neglected in the slow-roll situation when $|m^2|/H^2 \ll 1$.

D. Uniqueness of the solution

We have found there exists at least one regular solution that satisfies the adequate boundary condition which gives rise to slow-roll inflation on the brane, provided the model parameters are chosen such that $|m^2|/H^2 \ll 1$. We now ask if our solution is unique or not. If not, and if another solution happens to violate the slow-roll condition, inflation on the brane would not stably last.

To see if there is such a possibility, we first rewrite the radial eigen-value equation (3.14) in the standard Schrödinger form. To do so, we introduce the conformal radial coordinate η through $dr/R(r) = d\eta$, where $R(r) = \ell_{eff} \sinh(r/\ell_{eff})$. Then the metric (3.1) is expressed as

$$ds^2 = R^2 \left(d\eta^2 - H^2 dt^2 + \cosh^2(Ht)\, d\Omega_{(3)}^2 \right),$$ (3.35)

where

$$R(\eta) = \frac{\ell_{eff}}{\sinh(|\eta| + \eta_0)} \quad (-\infty < \eta < +\infty),$$ (3.36)

and η_0 is defined by the equation

$$\sinh(\eta_0) = \frac{1}{\sinh(r_0/\ell_{eff})} = H\ell_{eff}.$$ (3.37)

Then putting $u_\gamma = R^{-3/2} f(\eta)$, Eq. (3.14) becomes

$$-f'' + \tilde{V}f = -\lambda_\gamma^2 f,$$ (3.38)

8

where the prime denotes the η-derivative and

$$
\begin{aligned}
\tilde{V} &= \frac{(R^{3/2})''}{R^{3/2}} + m^2 R^2 \\
&= \frac{9}{4} + \frac{15 + 4m^2 \ell_{eff}^2}{4 \sinh^2(|\eta| + \eta_0)} - 3 \coth(|\eta| + \eta_0)\delta(\eta).
\end{aligned} \tag{3.39}
$$

For $m^2 \ell_{eff}^2 > -4$, this potential is of the volcano-type. Hence the solution we have found corresponds to the unique bound state solution with $E = -\lambda_\gamma^2 < 0$. Thus the solution is unique when $m^2 \ell_{eff}^2 > -4$.

On the other hand, when $m^2 \ell_{eff}^2 < -4$, our solution might not be unique, depending on the location of the brane. To investigate this case in more detail, it is useful to analyze the behavior of the function $u_\gamma(r)$ given by Eq. (3.21) by artificially putting $\gamma = 0$ (i.e., $\lambda_\gamma^2 = 0$). The function $u_0(r)$ can be expressed in terms of the elementary functions as

$$
u_0(r) = \frac{\sqrt{2/\pi}}{|\nu|(|\nu|^2 + 1)} \frac{\sin(|\nu|\, r/\ell_{eff}) - |\nu|\cos(|\nu|\, r/\ell_{eff})\tanh(r/\ell_{eff})}{\sinh^2(r/\ell_{eff})\tanh(r/\ell_{eff})}, \tag{3.40}
$$

where $|\nu| = \sqrt{|m^2|\ell_{eff}^2 - 4}$. We then immediately see that the first node appears in the region $\pi < |\nu| r/\ell_{eff} < 3\pi/2$. Thus, if the location of brane r_0 satisfies $r_0/\ell_{eff} < \pi/|\nu|$, u_0 has no node. For a large value of $|\nu|$, Eq. (3.12) with the requirement $|m^2|/H^2 \ll 1$ gives

$$
\left(\frac{r_0}{\ell_{eff}}\right)^2 (|\nu|^2 + 4) = \frac{|m^2|}{H^2}. \tag{3.41}
$$

Thus the above nodeless condition is safely satisfied. Since $u_0(r)$ corresponds to the zero energy solution, this implies there is no bound state solution in the energy range $0 > E > -\lambda_\gamma^2$. Hence our solution turns out to be unique also in this case.

IV. SUMMARY AND DISCUSSION

We have investigated the possiblity of brane-world inflation driven solely a bulk gravitational scalar field.

First we have clarified general (sufficient) conditions for inflation to occur on the brane by analyzing the effective 4-dimensional Einstein equations. Namely, we have found that the standard slow-roll inflation can occur if the bulk scalar field is sufficiently slowly varying both in space and time near the brane.

Then, we have modeled the effective potential of the gravitational scalar field by an new inflation type potential, and looked for a spherically symmetric 5-dimensional solution perturbatively. In our model, the lowest order solution is given by the AdS bulk with the de Sitter brane as the boundary, and non-trivial behavior of the scalar field appears at the first order. We have found there exists a regular solution for the scalar field in the separable form with respect to r and t, where r is the 5th dimensional radial coordinate. The solution we have obtained gives slow-roll inflation on the brane and is found to be unique, provided that $|m^2|/H^2 \ll 1$, which is the same condition for slow-roll inflation as in the usual 4-dimensional theory.

Although our solution is valid only perturbatively, we expect a regular spherically symmetric solution to exist as long as the scalar field potential is sufficiently flat. Then, an immediate question is whether our model can give the standard inflationary quantum fluctuation spectrum. In this respect, very recently, Kobayashi, Koyama, and Soda have calculated quantum fluctuations of a 5-dimensional minimally coupled massless scalar field, and concluded that the Kaluza-Klein modes give negligible contributions to the quantum fluctuation spectrum induced on the brane [42]. Hence we expect the same conclusion holds also in the present model.

However, the situation seems to change drastically at the end of inflation when the scalar field undergoes damped oscillations. As one can notice from the discussions given in Sec. III C, the radial function can be regular at the origin only if $m^2 < 0$. Thus there exists no regular solution in the separable form if $m^2 > 0$. This implies that any regular solution must be in the form, $\sum_\gamma \psi_\gamma(t)u_\gamma(r)$, where the sum extends over infinite numbers of γ. Then there seems no apparent reason why the sum should extend only over functions of t and r. In other words, it may be that an infinitesimal non-spherical perturbation (the so-called Kaluza-Klein mode) can grow indefinitely. This may cause a problem in our scenario. But at the same time, if the non-spherical perturbations saturate at certain level, this rather chaotic behavior may give rise to an efficient mechanism of reheating. Furthermore, the Kaluza-Klein excitations left in the subsequent universe may become a good candidate for the cold dark matter of the universe. Certainly, this issue deserves further studies.

Another remaining issue is the naturalness of the model parameters. If we assume $\kappa_5^2 V_0$ is of the same order of Λ_5, we have $\ell_{eff} \gtrsim \ell$. Since the location of the brane r_0 is expected to be greater than ℓ by factor of a few at least,

in order for our classical picture of the infinitely thin brane to be valid, r_0 cannot be smaller than ℓ_{eff}. Hence the condition $|m^2|/H^2 < 1$ implies $|m^2|\ell_{eff}^2 \ll 1$ (see Eq. (3.12)). That is, the mass parameter in our theory must be fine-tuned to a very small value compared with the natural mass scale of the theory. This is the same problem one encounters in 4-dimensional models of inflation. However, if we recall that our model corresponds to a conformally transformed scalar-tensor theory, this problem may be solved in the original conformal frame. Investigations in this direction is also left for future work.

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